d-exact categories and d-cluster tilting

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Reconstruction problem

$\mathcal{E}$ exact category, $M \in \mathcal{E}$ d-cluster tilting.
Can we determine $\mathcal{E}$ from $\operatorname{End}_{\mathcal{E}}(M)$?

Plan of the talk

1. Background.
2. Main results.
3. Idea of proofs.

Background

Background

Definition

A *conflation category* is a pair $(\mathcal{E}, \mathcal{S})$ where

- $\mathcal{E}$ is an additive category
- $\mathcal{S}$ is a class of kernel-cokernel pairs in $\mathcal{E}$, i.e. $\mathcal{S} \subset \{E_1 \xrightarrow{f} E_2 \xrightarrow{g} E_3 \mid f \text{ kernel of } g \text{ and } g \text{ cokernel of } f\}$. Assume $\mathcal{S}$ is closed under isomorphisms.

Elements of $\mathcal{S}$ are called *conflations*.

A conflation category is an additive category with an additional *structure*.

Exact categories

Idea: Inherit the conflations from abelian categories.

Definition

An *exact category* is a conflation category $(\mathcal{E}, \mathcal{S})$ where $\mathcal{E}$ is equivalent to a full *extension-closed* subcategory of an abelian category $\mathcal{A}$ such that $\mathcal{S}$ are the kernel-cokernel pairs in $\mathcal{E}$ which are sent to short exact sequences in $\mathcal{A}$.

Here $\mathcal{B} \subseteq \mathcal{A}$ extension-closed means if $0 \to A_1 \to A_2 \to A_3 \to 0$ short exact sequences in $\mathcal{A}$ with $A_1, A_3 \in \mathcal{B}$, then $A_2 \in \mathcal{B}$.
Can do homological algebra in $(\mathcal{E}, \mathcal{S})$ relative to $\mathcal{S}$. Leads to projective and injective objects, $\mathrm{Ext}^i_{\mathcal{E}}(-,-)$, derived categories...

Exact categories

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We often omit $\mathcal{S}$ and simply write $\mathcal{E}$ for $(\mathcal{E}, \mathcal{S})$.
A long exact sequence in $\mathcal{E}$ is a complex

$$\cdots \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow E_{i+1} \longrightarrow \cdots$$

Exact categories

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We often omit $\mathcal{S}$ and simply write $\mathcal{E}$ for $(\mathcal{E}, \mathcal{S})$.
A long exact sequence in $\mathcal{E}$ is a complex

$$\begin{tikzcd}
 & F_{i-1} \arrow[dr, dashed] & & & & F_{i+1} \arrow[dr, dashed] & & & \\
\cdots \arrow[rr] \arrow[ur, dashed] & & E_{i-1} \arrow[rr] \arrow[dr, dashed] & & E_i \arrow[rr] \arrow[ur, dashed] & & E_{i+1} \arrow[rr] \arrow[dr, dashed] & & \cdots \\
 & & & F_i \arrow[ur, dashed] & & & & F_{i+2} \arrow[ur, dashed] &
\end{tikzcd}$$

for which there exists objects F_i and dashed arrows such that each triangle commutes and each diagonal is a conflation.

d-cluster tilting

Fix an integer $d \geq 1$ and an exact category $\mathcal{E}$.

Definition (Iyama'06)

A full subcategory $\mathcal{M} \subset \mathcal{E}$ is *d-cluster tilting* if

- $\operatorname{Ext}^i_{\mathcal{E}}(M, M') = 0$ for all $0 < i < d$ and $M, M' \in \mathcal{M}$.
- For all $E \in \mathcal{E}$ there exists exact sequences in $\mathcal{E}$

$$0 \to M_d \to \cdots \to M_1 \to E \to 0$$
$$0 \to E \to N_1 \to \cdots \to N_d \to 0$$

  where $M_1, \ldots, M_d, N_1 \ldots, N_d \in \mathcal{M}$.
- $\mathcal{M}$ is closed under finite direct sums and summands.

$M \in \mathcal{E}$ is *d-cluster tilting* if add M is a d-cluster tilting subcategory of $\mathcal{E}$.

Note that $\mathcal{M} \subset \mathcal{E}$ is 1-cluster tilting if and only if $\mathcal{M} = \mathcal{E}$.
In particular, $M \in \mathcal{E}$ is 1-cluster tilting if and only if add $M = \mathcal{E}$.

Why?

d-cluster tilting subcategories are used in many different contexts.

- Proof of the Donovan Wemyss conjecture (Jasso–Keller–Muro'22).
- Partially wrapped Fukaya categories (Dyckerhoff–Jasso–Lekili'21).
- Silting complexes (via functorially finite higher torsion classes) (work in progress).
- NCCR's of isolated singularities.
- Higher dimensional McKay correspondence.
- Frobenius exact enhancements of cluster categories.

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d-cluster tilting in exact categories.

Main results

Reconstruction problem

$\mathcal{E}$ exact category, $M \in \mathcal{E}$ d-cluster tilting.
Can we determine $\mathcal{E}$ from $\mathrm{End}_{\mathcal{E}}(M)$? NO
M is always d-cluster tilting in add M (with split exact structure) but add $M \neq \mathcal{E}$ in general.
Need additional structure!

Theorem (Jasso'16)

Any d-cluster tilting subcategory of $\mathcal{E}$ is a d-exact category.

This is sufficient!

Theorem (K'25)

Any weakly idempotent complete d-exact category is d-cluster tilting in a unique (up to exact equivalence) exact category.

Weakly idempotent complete means all split epimorphisms have kernels, or equivalently, all split monomorphisms have cokernels.

Why?

Finding d-exact categories is often easier than finding d-cluster tilting subcategories.

- Any additive category has a maximal d-exact structure (Klapproth'24).
- Subfunctors of $\operatorname{Ext}^d(-,-)$ for d-cluster tilting subcategories (Hafezi–Asadollahi–Zhang'25)

The definition of d-exact categories generalizes the axiomatic characterization of exact categories. We first recall this.
Fix a conflation category $(\mathcal{E}, \mathcal{S})$. A morphism $f\colon E \to E'$ in $\mathcal{E}$ is a

- *inflation* if $(E \xrightarrow{f} E' \xrightarrow{g} E'') \in \mathcal{S}$ for some g.
- *deflation* if $(E'' \xrightarrow{g} E \xrightarrow{f} E') \in \mathcal{S}$ for some g.

Theorem (Quillen-Gabriel embedding theorem)

A conflation category $(\mathcal{E}, \mathcal{S})$ is an exact category if and only if

1. $(0 \to 0 \to 0) \in \mathcal{S}$.
2. *Inflations and deflations are closed under composition.*
3. *For all $(E_1 \xrightarrow{f} E_2 \to E_3) \in \mathcal{S}$ and morphisms $E_1 \xrightarrow{g} F_1$ there exists a commutative diagram*

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "g"] & E_2 \arrow[d, "h"] \\ F_1 \arrow[r, "k"] & F_2 \end{tikzcd}$$

where $(E_1 \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} E_2 \oplus F_1 \xrightarrow{\begin{pmatrix} h & -k \end{pmatrix}} F_2) \in \mathcal{S}$.

4. *Dual of (3).*

A d-exact category is defined roughly by replacing the short exact sequences with longer exact sequences.

Definition (Jasso'16)

Let $\mathcal{C}$ be an additive category. A *d-exact sequence* in $\mathcal{C}$ is a complex

$$X_0 \to \cdots \to X_{d+1}$$

in $\mathcal{C}$ such that

$$0 \to \mathcal{C}(X, X_0) \to \cdots \to \mathcal{C}(X, X_{d+1})$$

and

$$0 \to \mathcal{C}(X_{d+1}, X) \to \cdots \to \mathcal{C}(X_0, X)$$

are exact for all $X \in \mathcal{C}$.

Definition

A *d-conflation category* is a pair $(\mathcal{C}, \mathcal{X})$ where

- $\mathcal{C}$ is an additive category
- $\mathcal{X}$ is a class of d-exact sequences in $\mathcal{C}$, closed under *homotopy equivalence*.

The elements of $\mathcal{X}$ are called *admissible* d-exact sequences.

Fix a d-conflation category $(\mathcal{C}, \mathcal{X})$. A morphism $f\colon E \to E'$ in $\mathcal{C}$ is a

- *admissible monomorphism* if $(E \xrightarrow{f} E' \to E_2 \to \cdots \to E_{d+1}) \in \mathcal{X}$ for some complex $E' \to E_2 \to \cdots \to E_{d+1}$.

- *admissible epimorphism* if $(E_0 \to \cdots \to E_{d-1} \to E \xrightarrow{f} E') \in \mathcal{X}$ for some complex $E_0 \to \cdots \to E_{d-1} \to E$.

Theorem (Quillen-Gabriel embedding theorem)

A conflation category $(\mathcal{E}, \mathcal{S})$ is an exact category if and only if

1. $(0 \to 0 \to 0) \in \mathcal{S}$.
2. *Inflations and deflations are closed under composition.*
3. *For all $(E_1 \xrightarrow{f} E_2 \to E_3) \in \mathcal{S}$ and morphisms $E_1 \xrightarrow{g} F_1$ there exists a commutative diagram*

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "g"] & E_2 \arrow[d, "h"] \\ F_1 \arrow[r, "k"] & F_2 \end{tikzcd}$$

where $(E_1 \xrightarrow{\begin{pmatrix} f \\ g \end{pmatrix}} E_2 \oplus F_1 \xrightarrow{\begin{pmatrix} h & -k \end{pmatrix}} F_2) \in \mathcal{S}$.

4. *Dual of (3).*

Definition

A *d-conflation category* $(\mathcal{C}, \mathcal{X})$ is a *d-exact category* if and only if

1. $(0 \to 0 \to \cdots \to 0) \in \mathcal{X}$.
2. Admissible monomorphisms and admissible epimorphisms are closed under composition.
3. For all $(E_0 \to E_1 \to \cdots \to E_{d+1}) \in \mathcal{X}$ and morphisms $E_0 \to F_0$ there exists a morphism of complexes

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & \cdots \arrow[r] & E_d \arrow[d] \\ F_0 \arrow[r] & F_1 \arrow[r] & \cdots \arrow[r] & F_d \end{tikzcd}$$

   where the cone $(E_0 \to E_1 \oplus F_0 \to \cdots \to E_d \oplus F_{d-1} \to F_d)$ is in $\mathcal{X}$.
4. Dual of (3).

$d = 1$ recovers the notion of exact categories. We often omit $\mathcal{X}$ and simply write $\mathcal{C}$ for $(\mathcal{C}, \mathcal{X})$.

Theorem (Jasso'16)

Let $\mathcal{E}$ be a weakly idempotent complete exact category. Let $\mathcal{M}$ be a d-cluster tilting subcategory of $\mathcal{E}$. Then $(\mathcal{M}, \mathcal{X})$ is a d-exact category where $\mathcal{X}$ consists of all complexes in $\mathcal{M}$

$$0 \to X_0 \to X_1 \to \cdots \to X_{d+1} \to 0$$

which are exact in $\mathcal{E}$.

Theorem (K'25)

Let $\mathcal{M}$ be a weakly idempotent complete d-exact category. Then there exists an exact category $\mathcal{E}$ and a d-cluster tilting subcategory $\mathcal{N}$ of $\mathcal{E}$ such that $\mathcal{M} \cong \mathcal{N}$ as d-exact categories. Furthermore, $\mathcal{E}$ is unique up to exact equivalence.

Idea of proofs

Uniqueness

Theorem (K'25)

Let $\mathcal{M}$ be a weakly idempotent complete d-exact category. Then there exists an exact category $\mathcal{E}$ and a d-cluster tilting subcategory $\mathcal{N}$ of $\mathcal{E}$ such that $\mathcal{M} \cong \mathcal{N}$ as d-exact categories. Furthermore, $\mathcal{E}$ is unique up to exact equivalence.

Uniqueness follows from a universal property of the functor $\mathcal{M} \to \mathcal{E}$.

Definition

Let $\mathcal{M}$ be a d-exact category and $\mathcal{E}$ an exact category. A functor $F\colon \mathcal{M} \to \mathcal{E}$ is *exact* if whenever $(X_0 \to X_1 \to \cdots \to X_{d+1})$ is an admissible d-exact sequence in $\mathcal{M}$, then

$$0 \to F(X_0) \to F(X_1) \to \cdots \to F(X_{d+1}) \to 0$$

is exact in $\mathcal{E}$.

Uniqueness

Theorem (K'25)

Let $\mathcal{M}$ be a d-cluster tilting subcategory of an exact category $\mathcal{E}$. The following hold.

- *The inclusion $\mathcal{M} \to \mathcal{E}$ is exact.*
- *For any weakly idempotent complete exact category $\mathcal{E}'$ and exact functor $F\colon \mathcal{M} \to \mathcal{E}'$ there exists an exact functor $\mathcal{E} \to \mathcal{E}'$ extending F, unique up to isomorphism.*

$$\begin{tikzcd} \mathcal{E} \arrow[dr, dashed] & \\ \mathcal{M} \arrow[u] \arrow[r, "F"] & \mathcal{E}' \end{tikzcd}$$

Existence

How to show that any d-exact category is equivalent to a d-cluster tilting subcategory?
The following notion plays a central role.

Definition

Let $\mathcal{U}$ be a full subcategory of an exact category $\mathcal{E}$. We say that $\mathcal{U}$ is *d-extension closed* if any exact sequence

$$0 \to U \to E^1 \to \cdots \to E^d \to V \to 0$$

with $U, V \in \mathcal{U}$ is Yoneda equivalent to an exact sequence

$$0 \to U \to U^1 \to \cdots \to U^d \to V \to 0$$

with $U^1 \ldots, U^d \in \mathcal{U}$.

$d = 1$ correspond to normal extension closure, since Yoneda equivalent short exact sequences have isomorphic middle term.

Existence

Definition

Let $\mathcal{M}$ be a d-exact category.

- A functor $F\colon \mathcal{M}^{\mathrm{op}} \to \mathrm{Ab}$ is called *left exact* if for any admissible exact sequence $(X_0 \to X_1 \to \cdots \to X_{d+1})$ the sequence $0 \to F(X_{d+1}) \to F(X_d) \to F(X_{d-1})$ is left exact.
- $\mathcal{L}(\mathcal{M}) = \{F\colon \mathcal{M}^{\mathrm{op}} \to \mathrm{Ab} \mid F \text{ is left exact}\}$.

Any representable functor $\mathcal{M}(-,X)$ is left exact. Hence, the Yoneda embedding induces a functor $\mathcal{M} \to \mathcal{L}(\mathcal{M})$.

Theorem (Ebrahimi'21,Ebrahimi–Nasr-Isfahani'23)

Let $\mathcal{M}$ be a d-exact category. Then $\mathcal{L}(M)$ is abelian and the essential image of $\mathcal{M} \to \mathcal{L}(\mathcal{M})$ is d-rigid and closed under d-extensions.

Here d-rigid means $\mathrm{Ext}^i_{\mathcal{L}(\mathcal{M})}(\mathcal{M}(-,X), \mathcal{M}(-,Y)) = 0$ for all $X, Y \in \mathcal{M}$ and $0 < i < d$.

$d = 1$ recovers Quillen-Gabriel embedding theorem.

Existence

Theorem (Ebrahimi'21,Ebrahimi–Nasr-Isfahani'23)

Let $\mathcal{M}$ be a d-exact category. Then $\mathcal{L}(M)$ is abelian and the essential image of $\mathcal{M} \to \mathcal{L}(\mathcal{M})$ is d-rigid and closed under d-extensions.

We show

Theorem (K'25)

Let $\mathcal{E}$ be an exact category, and let $\mathcal{U}$ be a weakly idempotent complete full subcategory of $\mathcal{E}$. Assume $\mathcal{U}$ is d-rigid, and d-extension-closed. Then there exists a unique extension closed subcategory $\mathcal{F}$ of $\mathcal{E}$ such that

- *$\mathcal{U}$ is a d-cluster tilting subcategory of $\mathcal{F}$.*
- *The map $\mathrm{Ext}^d_{\mathcal{F}}(U, V) \to \mathrm{Ext}^d_{\mathcal{E}}(U, V)$ is an isomorphism for all $U, V \in \mathcal{U}$.*

Combining these two results we get that any weakly idempotent complete d-exact category is equivalent to a d-cluster tilting subcategory.

Existence

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Let $\mathcal{E}$ be an exact category, and let $\mathcal{U}$ be a weakly idempotent complete full subcategory of $\mathcal{E}$. Assume $\mathcal{U}$ is d-rigid, and d-extension-closed. Then there exists a unique extension closed subcategory $\mathcal{F}$ of $\mathcal{E}$ such that

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Strategy: define

$$\mathcal{U}^d(\mathcal{E}) = \{E \in \mathcal{E} \mid \exists\ 0 \to E \to U^1 \to \cdots \to U^d \to 0 \text{ exact where } U^i \in \mathcal{U}\ \forall i\}$$

$$\mathcal{U}_d(\mathcal{E}) = \{E \in \mathcal{E} \mid \exists\ 0 \to U_d \to \cdots \to U_1 \to E \to 0 \text{ exact where } U_i \in \mathcal{U}\ \forall i\}$$

$\mathcal{U} \subset \mathcal{E}$ being d-rigid and d-extension closed implies $\mathcal{U}^d(\mathcal{E})$ and $\mathcal{U}_d(\mathcal{E})$ are extension closed and $\mathcal{U}$ is d-rigid and d-extension closed in $\mathcal{U}^d(\mathcal{E})$ and $\mathcal{U}_d(\mathcal{E})$.

Theorem (K'25)

Let $\mathcal{E}$ be an exact category, and let $\mathcal{U}$ be a full subcategory of $\mathcal{E}$ which is weakly idempotent complete, d-rigid, and d-extension-closed. Then there exists a unique extension closed subcategory $\mathcal{F}$ of $\mathcal{E}$ such that

- *$\mathcal{U}$ is a d-cluster tilting subcategory of $\mathcal{F}$.*
- *The map $\operatorname{Ext}^d_{\mathcal{F}}(U,V) \to \operatorname{Ext}^d_{\mathcal{E}}(U,V)$ is an isomorphism for all $U, V \in \mathcal{U}$.*

Can iterate the construction of $\mathcal{U}^d(\mathcal{E})$ and $\mathcal{U}_d(\mathcal{E})$ to get a sequence of subcategories

$$\cdots \subseteq \mathcal{U}^d\mathcal{U}_d\mathcal{U}^d(\mathcal{E}) \subseteq \mathcal{U}_d\mathcal{U}^d(\mathcal{E}) \subseteq \mathcal{U}^d(\mathcal{E}) \subseteq \mathcal{E}.$$

We show that this sequence stabilizes after two steps and furthermore

$$\mathcal{U}^d\mathcal{U}_d(\mathcal{E}) = \mathcal{U}_d\mathcal{U}^d(\mathcal{E}).$$

This is the extension closed subcategory $\mathcal{F}$ that we wanted!